A LYAPUNOV FUNCTION APPROACH TO ENERGY BASED MODEL REDUCTION

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ABSTRACT

Model reduction based upon the idea of eliminating coordinates with low levels of associated power, energy or activity has been proposed by a number of researchers. None of these results, however, produce the sort of computable bounds on the neglected dynamics that would be useful in the design of controllers with guaranteed robustness properties. This paper outlines an approach to model reduction based upon Lyapunov functions that represent a modified version of the system energy of Lagrangian subsystems. The Lyapunov functions are used to bound the states of subsystems to be removed, enabling these states to be treated as time-varying perturbations in a simplified set of dynamic equations. In contrast to other results in energy-based model reduction, this approach provides bounds on the disturbances caused by the unmodeled dynamics though at the cost of the implementation ease associated with other methods.

1 INTRODUCTION — REDUCTION OF PROPER MODELS

In the design of mechanical and mechatronic systems, dynamic models with physically signficant state variables and parameters offer distinct advantages over more empirical or identified models. Dubbed "proper models" by Wilson and Stein [1], these representations tie the dynamic behavior of a complex system directly to design parameters, thus enabling simultaneous treatment of cost, complexity and performance issues before prototyping and identification occur. Proper models can also play an important role in "Design for Controllability" — the process of identifying control issues and designing preliminary controllers early in the design cycle. A challenge, however, stems from the fact that proper models developed for design (through CAD and multi-body dynamic modeling, for instance) are much more complex than those required for control. Model reduction is therefore necessary.

A number of previous researchers have developed techniques for reducing proper models based on metrics related to energy or power. Rosenberg and Zhou [2] suggested using power to evaluate what bonds could be removed from a bond graph model. Loucas and Stein [3] discussed the merits of activity — the time integral of the absolute value of power. This value could be easily computed for bond graphs, multi-body dynamics [4] and linear systems [5]. Yi and Youcef-Toumi [6] proposed a similar idea for reducing bond graph models by comparing the energy associated with each bond to those in neighboring bonds and eliminating those with smallest relative energy.

Under all of these methods, model reduction consists of running a series of simulations, ranking the individual coordinates or elements by the appropiate metric and removing those that fall below a certain threshold. This approach is well-suited for complex models such as those encountered in multi-body dynamics and, not surprisingly, has proven successful in such applications [4]. From the standpoint of control, however, these methods suffer from the fact that they do not provide any bounds for the effects of the neglected dynamics. Work in the past decade in the area of robust nonlinear control has produced a number of methods — such as backstepping [7] and dynamic surface control [8] — for providing bounded tracking or regulation of uncertain nonlinear systems, provided the uncertainty can be bounded. A model reduction technique that could produce bounds on the effect of the neglected dynamics on the remaining dynamics (for some bounded region of the state space of the remaining dynamics) would therefore be of great use in modeling for control.

This paper outlines an approach to model reduction that

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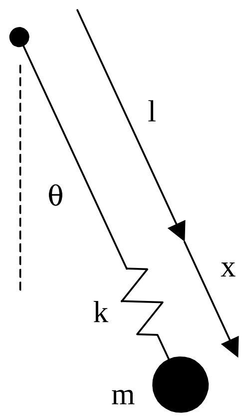

Figure 1: The Swinging Spring Model

achieves such a bound. Like the methods described above, it is fundamentally related to the idea of eliminating coordinates with low associated energy. The method covers systems of Lagrangian dynamics where Lagrangian subsystems are connected by (generally state-dependent) forces. By modifying the energy of the coordinates to be eliminated, a Lyapunov function can be obtained and subsequently used to establish bounds on these coordinates. The system can then be represented by a reduced set of equations with the neglected states replaced by their bounds. The basic intuition is similar to other work in the area of proper model reduction in the sense that small values of the Lyapunov function (like energy) denote candidates for reduction.

Section 2 explains the approach via the simple example of a swinging spring. Section 3 continues this example by analyzing a Lyapunov function used to obtain a bound for the neglected states. Sections 4 and 5 extends these ideas to more general systems.

2 PROBLEM STATEMENT AND ILLUSTRATIVE EXAMPLE

In this work, we restrict attention to Lagrangian systems that are comprised of Lagrangian subsystems with force interconnections. An example is the swinging spring in Figure 1 which is comprised of a pendulum mode and a spring mode - both of which independently are Lagrangian systems. These two systems are connected by a force with a definite physical interpretation: it represents the constraint force associated with the simple single degree-of-freedom model of the pendulum (in other words, the case of an infinitely stiff spring). Thus, the model reduction goal of eliminating the spring and treating the system as a pure pendulum has the natural interpretation of replacing a degree of freedom with a workless constraint.

In terms of equations, the kinetic energy, T, and potential energy, V, of this system are given by:

$$T = \frac{1}{2}m\left(l+x\right)^2\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2 \tag{1}$$

$$V = \frac{1}{2}kx^2 + V_0 - mg\left(l+x\right)\cos\theta \tag{2}$$

where V_0 is an offset to give positive potential energy at a reference height. This is clearly a Lagrangian system and the equations of motion are given by

$$\left(l+x\right)\ddot{\theta} + 2\dot{x}\dot{\theta} + g\sin\theta = u(t) \tag{3}$$

$$m\ddot{x} + b\dot{x} + kx = m\left(l+x\right)\dot{\theta}^2 - mg\cos\theta \tag{4}$$

if we assume a control torque $u(t)$ applied around the axis of rotation and additional damping $b\dot{x}$ in the spring (or in general, Rayleigh dissipation). The equation of motion of the proposed model, a standard single degree of freedom pendulum, is

$$\ddot{\theta} = -\frac{1}{l}g\sin\theta + \frac{1}{l}u. \tag{5}$$

Writing (3) more suggestively,

$$\ddot{\theta} = -\frac{1}{l}g\sin\theta + \Delta_1 + (\frac{1}{l} + \Delta_2)u \tag{6}$$

$$\begin{aligned} \Delta_1 &= \frac{x}{l(l+x)}g\sin\theta - \frac{2\dot{x}\dot{\theta}}{l+x} \\ \Delta_2 &= -\frac{x}{l+x} \end{aligned} \tag{7}$$

The reduction problem examined here is to find a range of operation for the pendulum states in which the unmodeled terms Δ_1 and Δ_2 can be treated as bounded disturbances in the sense that $|\Delta_1| \le d_1(\theta,\dot{\theta})$ and $|\Delta_2| \le d_2(\theta,\dot{\theta})$ for some known functions d_1 and d_2. This can be achieved if the spring states $(x,\dot{x})$ can be bounded. The work on robust control can then be brought to bear on designing u.

Interpreting (4) as a Lagrangian system driven by an input force, the following section uses a Lyapunov function derived from the system energy to compute a bound on $(x,\dot{x})$ by establishing bounded input bounded state stability.

3 LINEAR SYSTEM ANALYSIS

3.1 Selection of a Suitable Lyapunov Function

Since the spring mode is a simple linear mass-spring-damper system (Figure 2), we begin the analysis with this case. The unforced equations take the following form:

$$m\ddot{x} + b\dot{x} + kx = 0 \tag{8}$$

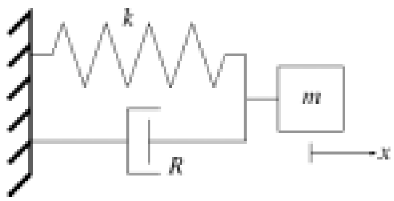

Figure 2: Classical spring mass damper system.

Bounded input bounded state stability can be derived from a positive definite Lyapunov function with negative definite derivative [9] and the total energy of the system provides a logical starting point in the search for Lyapunov functions. Let E be the total energy of the system.

$$E = V + T = \frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2 \tag{9}$$

$$= \frac{1}{2}\begin{bmatrix} x & \dot{x} \end{bmatrix} \begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \tag{10}$$

Now take the derivative with respect to time and substitute from (8).

$$\dot{E} = kx\dot{x} + m\dot{x}\ddot{x} \tag{11}$$

$$= kx\dot{x} - m\dot{x}(\frac{b\dot{x} + kx}{m}) = -b\dot{x}^2 \tag{12}$$

$\dot{E}$ in this case is negative semidefinite. At the extremes of motion, $x \neq 0$, $\dot{x} = 0 \Rightarrow \dot{E} = 0$, which is a non-equilibrium position. This makes intuitive sense since the damper can only remove energy from the system when the velocity term is non-zero. Figure 3 illustrates this well-known result. Clearly, energy does not provide the negative definite derivative that could be used to bound the states.

So if energy is not a suitable Lyapunov function, what is? Converse Lyapunov theory tells us that if a system is exponentially stable, then a positive definite function with a negative definite derivative must exist. For the mass spring damper system, such functions may be found by introducing a cross term. Let

$$W = V + T + \varepsilon x\dot{x} = \frac{1}{2}\begin{bmatrix} x & \dot{x} \end{bmatrix} \begin{bmatrix} k & \varepsilon \\ \varepsilon & m \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}. \tag{13}$$

By Sylvester's criterion, W is positive definite if

$$0 < \varepsilon < \sqrt{km}. \tag{14}$$

Taking the derivative of W and substituting (8),

$$\dot{W} = m\dot{x}\ddot{x} + kx\dot{x} + \varepsilon\dot{x}^2 + \varepsilon x\ddot{x} \tag{15}$$

$$= (\varepsilon - b)\dot{x}^2 - \frac{\varepsilon}{m}bx\dot{x} - \frac{\varepsilon}{m}kx^2 \tag{16}$$

$$= -\begin{bmatrix} x & \dot{x} \end{bmatrix} \begin{bmatrix} \frac{\varepsilon k}{m} & \frac{\varepsilon b}{2m} \\ \frac{\varepsilon b}{2m} & (b - \varepsilon) \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}. \tag{17}$$

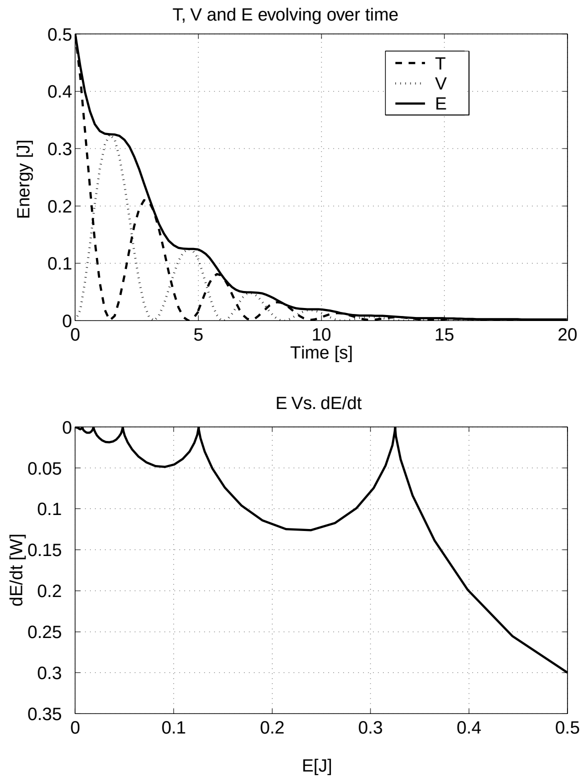

Figure 3: Top: Kinetic, potential, and total energy vs. time; Bottom: $\dot{E}$ vs. E — the semi-definiteness of $\dot{E}$ is apparent from the zeros.

Then $\dot{W}$ is negative definite if

$$0 < \varepsilon < \frac{4kbm}{b^2 + 4km}. \tag{18}$$

Thus, there exists a range of ε which yields a negative definite $\dot{W}$ and a positive definite W. Indeed, an exponential decay for W is possible by choosing ε to enforce $\dot{W} = -\alpha W$. From (13) and (17), this results in $\varepsilon = \frac{b}{2}$, $\alpha = \frac{b}{m}$. With this choice of ε, $k > \frac{b^2}{4m}$ ensures (14) and (18).

The resulting Lyapunov function decays exponentially (see Figure 4), as desired.

3.2 Bounded Input Bounded State Stability

Analyzing the Lyapunov function introduced above also yields an estimate of the states for a bounded input force. The forced mass-spring-damper system evolves according to

$$m\ddot{x}(t) = -kx(t) - b\dot{x}(t) + u(t). \tag{19}$$

Suppose that for some $U \in \mathbb{R}$, $|u(t)| \leq U$ for all t. Taking the time derivative of $W(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 + \frac{b}{2}x\dot{x}$ and

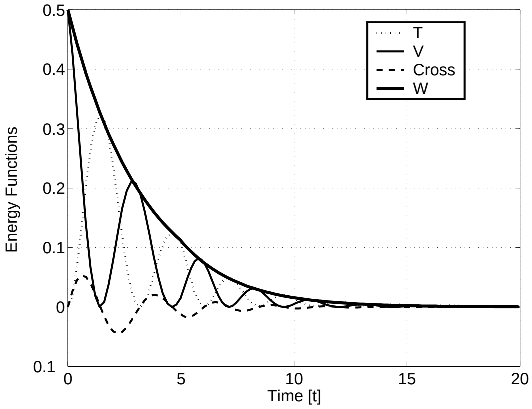

Figure 4: Kinetic and potential energy and W vs. time

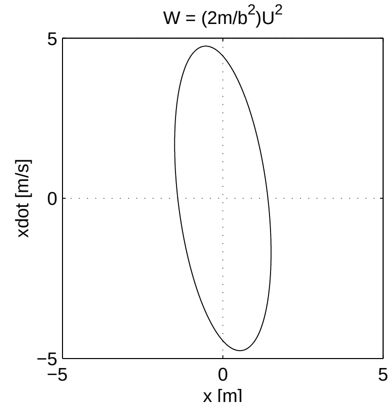

Figure 5: Maximum state deviation for a bounded input. $m = 1$ kg, $k = 10$ N/m, $b = 4.5$ N/(m/s), $U = 10$ N

substituting (19),

$$\frac{d}{dt}W = -\frac{b}{m}W + \Big(\dot{x} + \frac{b}{2m}x\Big)u.$$

Thus,

$$2\sqrt{W}\frac{d}{dt}\sqrt{W} \leq -\frac{b}{m}W + U\sqrt{\dot{x}^2 + \frac{b^2}{4m^2}x^2 + \frac{b}{m}x\dot{x}}. \quad (20)$$

Dividing by $\sqrt{W}$ and recalling the requirement that $k > \frac{b^2}{4m}$,

$$\frac{d}{dt}\sqrt{W} \leq -\frac{b}{2m}\sqrt{W} + U\frac{1}{2}\sqrt{\frac{2}{m}} \quad (21)$$

$$\Rightarrow \sqrt{W} \leq e^{-\frac{b}{2m}t}\Big(\sqrt{W_0} - \frac{\sqrt{2m}}{b}U\Big) + \frac{\sqrt{2m}}{b}U. \quad (22)$$

Then

$$\begin{aligned} W(t) \leq{}& e^{-\frac{b}{m}t}\Big(\sqrt{W_0} - \frac{\sqrt{2m}}{b}U\Big)^2 \\ &+ e^{-\frac{b}{2m}t}\Big(\sqrt{W_0} - \frac{\sqrt{2m}}{b}U\Big)\frac{\sqrt{2m}}{b}U \\ &+ \frac{2m}{b^2}U^2. \end{aligned} \quad (23)$$

The states can then be estimated by the level curve of W corresponding to $\frac{2m}{b^2}U^2$, as illustrated in Figure 5.

3.3 Bounding the Unmodeled States

Focusing again on bounding the unmodeled states from the spring degree of freedom of the swinging pendulum of §2, the preceding computation shows that if $W_0 \leq \frac{2m}{b^2}U^2$ and $|m(l+x)\dot{\theta}^2 - mg\cos\theta| \leq U$ whenever $W(x,\dot{x}) \leq \frac{2m}{b^2}U^2$, then $W(x,\dot{x}) \leq \frac{2m}{b^2}U^2$ for all t. Observe that $x^2 \leq \frac{8m}{4km-b^2}W$ (see Lemma 1 in §5). It is easy to confirm that if

$$\dot{\theta}^2 \leq \frac{U - mg}{m(l + \frac{4m}{b\sqrt{4km-b^2}}U)} \quad (24)$$

then $|m(l+x)\dot{\theta}^2 - mg\cos\theta| \leq U$ whenever $W \leq \frac{2m}{b^2}U^2$.

Thus, (24) gives a restriction of the state space of the modeled system where the perturbations from $(x,\dot{x})$ are bounded.

4 EXTENSION TO MORE GENERAL SYSTEMS

The intuition from the preceding sections treating the spring-pendulum system can be extended to a more general class of Lagrangian systems. Following the paradigm of using "proper models", this paper addresses simplified models derived from a physically motivated truncation of the states. The goal is to produce a reduced order model which is a Lagrangian system subject to bounded disturbances, to which techniques from robust control can be applied.

Consider a system with configuration space $Q_1 \times Q_2 \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ whose Lagrangian admits the following decomposition (as presented in [10]):

$$L(q_1,q_2,\dot{q}_1,\dot{q}_2) = L_1(q_1,\dot{q}_1,q_2) + L_2(q_2,\dot{q}_2), \quad (25)$$

and has the usual structure

$$\begin{aligned} L(q_1,q_2,\dot{q}_1,\dot{q}_2) = \frac{1}{2}\begin{bmatrix}\dot{q}_1^T & \dot{q}_2^T\end{bmatrix} M(q_1,q_2)\begin{bmatrix}\dot{q}_1 \\ \dot{q}_2\end{bmatrix} \\ - V(q_1,q_2). \end{aligned} \quad (26)$$

If $L_2(q_2,\dot{q}_2) = \frac{1}{2}\dot{q}_2^T M_2(q_2)\dot{q}_2 - V_2(q_2)$, then (25) and (26) imply

$$M(q_1,q_2) = \begin{bmatrix} M_1(q_1,q_2) & 0 \\ 0 & M_2(q_2)\end{bmatrix}. \quad (27)$$

Suppose that the Q_1 directions are actuated and that the damping force is decoupled so that the equations of motion are

$$M(q_1,q_2)\begin{bmatrix}\ddot q_1\\ \ddot q_2\end{bmatrix}+C(q_1,q_2,\dot q_1,\dot q_2)\begin{bmatrix}\dot q_1\\ \dot q_2\end{bmatrix} + DV(q_1,q_2)+\begin{bmatrix}\phi_1(\dot q_1)\\ \phi_2(\dot q_2)\end{bmatrix}=\begin{bmatrix}u\\ 0\end{bmatrix}, \quad (28)$$

where DV denotes the derivative of V and $C(x,\dot x)$ is the matrix of Coriolis and centrifugal forces defined by $M(x)$ in the standard way via

$$C_{ij}=\frac{1}{2}\sum_{k=1}^{n}\left(\frac{\partial M_{ki}}{\partial x_j}+\frac{\partial M_{ij}}{\partial x_k}-\frac{\partial M_{jk}}{\partial x_i}\right)\dot x_k. \quad (29)$$

Assume V_2 is at a minimum when $q_2=0$. When the q_1 and q_2 directions are decoupled in this way and there is a restoring force pushing q_2 to 0, it makes sense to simplify equation (28) by fixing q_2 at 0 and re-deriving the dynamics.

Suppose the system is modeled with dynamics on Q_1 derived from the simplified Lagrangian

$$\hat L(q,\dot q)=L(q,0,\dot q,0)=\frac{1}{2}\dot q^T\hat M(q)\dot q-\hat V(q). \quad (30)$$

That is, the system is modeled by

$$\ddot q=-\hat N(q)\big(\hat C(q,\dot q)\dot q+D\hat V(q)+\phi_1(\dot q)\big)+\hat N(q)u, \quad (31)$$

where $\hat N(q)$ is the inverse of $\hat M(q)$, and $\hat C$ is the matrix representing the Coriolis and centrifugal forces associated with $\hat M$. Let C_2 represent the Coriolis and centrifugal forces associated with M_2, and make the following definitions:

$$\begin{aligned}\tilde N(q_1,q_2)&=M_1(q_1,q_2)^{-1}-\hat N(q_1)\\ \tilde V(q_1,q_2)&=V(q_1,q_2)-\hat V(q_1)\\ D_2\tilde V(q_1,q_2)&=\frac{\partial\tilde V}{\partial q_2}(q_1,q_2)\\ \begin{bmatrix}\tilde C_{11}&\tilde C_{12}\\ \tilde C_{21}&\tilde C_{22}\end{bmatrix}(q_1,q_2,\dot q_1,\dot q_2)&=C(q_1,q_2,\dot q_1,\dot q_2)\\ &\quad-\begin{bmatrix}\hat C(q_1,\dot q_1)&0\\ 0&C_2(q_2,\dot q_2)\end{bmatrix}.\end{aligned}$$

Observe that $\tilde C_{11}=\sum_{i=1}^{n_2}\frac{\partial M_2}{\partial q_{2i}}\dot q_{2i}$, and $\tilde C_{22}=0$. Equation (28) can then be written

$$\begin{aligned}\ddot q_1=&-\hat N(\hat C\dot q_1+D\hat V+\phi_1)\\ &-\tilde N(\hat C\dot q_1+D\hat V+\phi_1)\\ &-(\hat N+\tilde N)(\tilde C_{11}\dot q_1+\tilde C_{12}\dot q_2+D_1\tilde V)\\ &+(\hat N+\tilde N)u\end{aligned} \quad (32)$$

$$M_2\ddot q_2+C_2\dot q_2+DV_2+\phi_2=-\tilde C_{21}\dot q_1-D_2\tilde V. \quad (33)$$

Since $\tilde N(q_1,0)=\tilde C_{11}(q_1,0,\dot q_1,0)=D_1\tilde V(q_1,0)=0$, equation (32) is the same as equation (31) when $(q_2,\dot q_2)=0$.

As before, the total system decomposes into two systems — a Lagrangian system on Q_2 driven by a disturbance force (32), and the modeled system on Q_1 subject to perturbations from q_2 and $\dot q_2$ (33). Given a bound on the magnitude of $(q_2,\dot q_2)$, the system can be modeled as

$$\begin{aligned}\ddot q=-\hat N(q)\big(\hat C(q,\dot q)\dot q+D\hat V(q)+\phi_1(\dot q)\big)+\Delta_1\\ +(\hat N(q)+\Delta_2)u\end{aligned} \quad (34)$$

$$\|\Delta_1\|\le d_1(q,\dot q) \quad (35)$$

$$\|\Delta_2\|\le d_2(q,\dot q), \quad (36)$$

and robust control techniques can be used to design u. The following section gives one way to compute a bound on $(q_2,\dot q_2)$ given a bound on the right hand side of (33), and describes how this can be used with a restriction of Q_1 to establish a bound on the magnitude of $(q_2,\dot q_2)$.

5 STABILITY OF DAMPED MECHANICAL SYSTEMS

The natural stability of damped Lagrangian systems allows for the computation of a bound on the states given a bound on the input. This section examines the stability of a class of mechanical systems and determines a bound on the total energy given a bound on the input forcing by analyzing an energy-like Lyapunov function.

5.1 Preliminaries

Consider a mechanical system on a configuration space $Q\subset\mathbb{R}^n$, with kinetic energy $T(x,\xi)$, potential energy $V(x)$, damping $\phi(\xi,t)$, and forcing $u(t)$. Assume that $T(x,\xi)=\frac{1}{2}\xi^TM(x)\xi$, where the mass matrix $M(x)\in\mathbb{R}^{n\times n}$ is positive definite for each $x\in Q$. Then the trajectories $q(t)$ evolve according to the following equations of motion:

$$M(q)\ddot q+C(q,\dot q)\dot q+DV(q)+\phi(\dot q,t)=u(t), \quad (37)$$

where C is defined as in (29).

Let $A(x)$ be the symmetrix matrix given by

$$\begin{aligned}A_{ij}(x)=\frac{1}{2}\sum_{k=1}^{n}\Bigg(&\frac{\partial^2V}{\partial x_k\partial x_i}(x)M_{jk}(x)\\ &+\frac{\partial^2V}{\partial x_j\partial x_k}(x)M_{ki}(x)\\ &-\frac{\partial V}{\partial x_k}(x)\frac{\partial M_{ij}}{\partial x_k}(x)\Bigg).\end{aligned} \quad (38)$$

Straightforward computations yield the following two useful relationships between M, V, C, and A. First, $\dot M-2C$ is skew symmetric, and hence

$$\dot q^T\left(\frac{d}{dt}M(q(t))-2C(q,\dot q)\right)\dot q=0. \quad (39)$$

Second,

$$\dot{q}^T A(q)\dot{q} = \dot{q}^T D^2 V(q) M(q)\dot{q} + DV(q)^T \left(\frac{d}{dt}M(q) - C(q,\dot{q})\right)\dot{q}. \quad (40)$$

The first relationship is well-known (see [10]), and the second is the motivation for the definition of A.

If $T \in \mathbb{R}^{k\times k}$, let $\lambda_i(T)$ denote the ith largest real eigenvalue of T. The following lemma will facilitate comparisons of quadratic functions on $\mathbb{R}^2$ based on eigenvalues of matrices in $\mathbb{R}^{2\times 2}$.

Lemma 1 *Suppose $P, Q \in \mathbb{R}^{2\times 2}$ are symmetric, and P is positive definite. Then the eigenvalues of $P^{-1}Q$ are real and*

$$\min_{z\in\mathbb{R}^2, z\neq 0} \frac{z^T Q z}{z^T P z} = \lambda_2(P^{-1}Q)$$
$$\max_{z\in\mathbb{R}^2, z\neq 0} \frac{z^T Q z}{z^T P z} = \lambda_1(P^{-1}Q).$$

The lemma follows from the existence of a minimum and a maximum and the fact that any critical value of $\frac{z^TQz}{z^TPz}$ must be an eigenvalue of $P^{-1}Q$; the proof is omitted.

For $x \in Q$ and $v \in \mathbb{R}^n$, let $\|v\|$ be the usual norm on $\mathbb{R}^n$, and define $\|v\|_x$ by $\|v\|_x^2 = v^T M(x) v$ (that is, $\|\ \|_x$ is the norm on $\mathbb{R}^n$ induced by the positive definite matrix $M(x)$).

The ensuing discussion treats mechanical systems for which there exist positive real numbers μ_1, μ_2, γ_1, γ_2, a, and b such that, for all $x \in Q$, $\xi \in \mathbb{R}^n$, and $s \in \mathbb{R}$,

$$\mu_1\|\xi\|^2 \leq \|\xi\|_x^2 \leq \mu_2\|\xi\|^2, \quad (41)$$
$$\gamma_1 V(x) \leq \|DV(x)\|_x^2 \leq \gamma_2 V(x), \quad (42)$$
$$V(x) \geq 0, \quad (43)$$
$$a\|\xi\|^2 \geq \xi^T A(x)\xi, \quad (44)$$
$$b\|\xi\|^2 \leq \xi^T \phi(\xi, s). \quad (45)$$

Roughly speaking, (41) demands that the inertia does not vanish and does not blow up. Assumption (42) requires the potential to be quadratic-like; it allows for V to estimate $\|DV\|_x$. Indeed, if the potential V is quadratic in x, (42) is automatically satisfied. The requirement (45) on ϕ is the familiar fully damped assumption. Since Q can usually be taken to be compact, (41) and (44) are fairly innocuous.

5.2 Bounded Input Bounded State Stability

It is well known that the minima of the potential of mechanical systems described in §5.1 are asymptotically stable (see, for instance, [10]). If the mechanical system satisfies the assumptions in §5.1, a bounded input results in bounded total energy. The proposition below computes a bound for the total energy given a bound on $u(t)$. The proof follows the approach of Bullo and Murray in [11].

Given the parameters μ_1, μ_2, γ_1, γ_2, a, and b from (41) – (45), set $\epsilon_{\max} = \min\{\sqrt{\frac{2}{\gamma_2}}, \frac{4b}{b^2+4a}\}$. For any $0 < \epsilon < \epsilon_{\max}$, define the following symmetric matrices:

$$P_{1,\epsilon} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}\epsilon \\ -\frac{1}{2}\epsilon & \frac{1}{\gamma_2} \end{bmatrix} \quad (46)$$

$$P_{2,\epsilon} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}\epsilon \\ \frac{1}{2}\epsilon & \frac{1}{\gamma_1} \end{bmatrix} \quad (47)$$

$$Q_\epsilon = \begin{bmatrix} b - \epsilon a & -\frac{1}{2}\epsilon b \\ -\frac{1}{2}\epsilon b & \epsilon \end{bmatrix}. \quad (48)$$

Since $0 < \epsilon < \epsilon_{\max}$, $P_{1,\epsilon}$, $P_{2,\epsilon}$, and Q_ϵ are all positive definite. In addition, let

$$\eta_{1,\epsilon} = \lambda_2\left(P_{2,\epsilon}^{-1}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{\gamma_2}\end{bmatrix}\right)$$
$$\eta_{2,\epsilon} = \lambda_1\left(P_{1,\epsilon}^{-1}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{\gamma_1}\end{bmatrix}\right), \quad (49)$$

and

$$\sigma_\epsilon = \frac{1}{\mu_2}\lambda_2(P_{2,\epsilon}^{-1}Q_\epsilon). \quad (50)$$

Proposition 1 *Suppose there is a $U \in \mathbb{R}$ such that $\|u(t)\| \leq U$ for all $t \in \mathbb{R}$. Let*

$$K = \min_{0<\epsilon<\epsilon_{\max}} \frac{\eta_{2,\epsilon}(1+\epsilon^2)}{\sigma_\epsilon^2 \mu_1 \lambda_2(P_{2,\epsilon})}.$$

Then there are constants $c_1, c_2, \sigma \in \mathbb{R}$ such that

$$E(t) \leq c_1 e^{-\sigma t} + c_2 e^{-\frac{\sigma}{2}t} + KU^2.$$

In particular, for any $\delta > 0$, there is a time T such that $E(t) \leq KU^2 + \delta$ for all $t > T$ (provided that the solution $q(t)$ to the equations of motion (37) exists and is in Q).

Proof For some fixed ϵ, $0 < \epsilon < \epsilon_{\max}$, let

$$W(x,\xi) = \frac{1}{2}\|\xi\|_x^2 + V(x) + \epsilon\langle \xi, DV(x)\rangle_x, \quad (51)$$

where $\langle v_1, v_2\rangle_x = v_1^T M(x) v_2$, for $v_1, v_2 \in \mathbb{R}^n$, and $x \in Q$ (that is, $\langle\ ,\ \rangle_x$ is the inner product on $\mathbb{R}^n$ induced by the norm $\|\ \|_x$). Then (41) and (42) (together with the Cauchy-Schwarz inequality) yields

$$W \geq \mu_1 \begin{bmatrix}\|\xi\| & \|DV(x)\|\end{bmatrix} P_{1,\epsilon}\begin{bmatrix}\|\xi\| \\ \|DV(x)\|\end{bmatrix}, \quad (52)$$

$$W \leq \mu_2 \begin{bmatrix}\|\xi\| & \|DV(x)\|\end{bmatrix} P_{2,\epsilon}\begin{bmatrix}\|\xi\| \\ \|DV(x)\|\end{bmatrix}. \quad (53)$$

Taking the time derivative,

$$\begin{aligned}\frac{d}{dt}W(q(t),\dot q(t)) &= \dot q^T M(q)\ddot q + \frac{1}{2}\dot q^T \frac{d}{dt}M(q)\dot q \\ &\quad + \dot q^T DV(q) + \epsilon DV(q)^T M(q)\ddot q \\ &\quad + \epsilon DV(q)^T \frac{d}{dt}M(q)\dot q \\ &\quad + \epsilon \dot q^T D^2V(x)M(q)\dot q.\end{aligned}$$

Using equations (37), (39), and (40),

$$\begin{aligned}\frac{d}{dt}W &= -\dot q^T\phi(\dot q,t) - \dot q^T u(t) - \epsilon DV(q)^T DV(q) \\ &\quad - \epsilon DV(q)^T\phi(\dot q,t) - \epsilon DV(q)^T u(t) \\ &\quad + \epsilon \dot q^T A(q)\dot q.\end{aligned}$$

From (44) and (45),

$$\begin{aligned}\frac{d}{dt}W \le &- \begin{bmatrix}\|\dot q\| & \|DV(q)\|\end{bmatrix} Q_\epsilon \begin{bmatrix}\|\dot q\| \\ \|DV(q)\|\end{bmatrix} \\ &+ \|u(t)\| \begin{bmatrix}1 & \epsilon\end{bmatrix}\begin{bmatrix}\|\dot q\| \\ \|DV(q)\|\end{bmatrix}.\end{aligned} \tag{54}$$

Then from (52) and $\frac{d}{dt}W = 2\sqrt{W}\frac{d}{dt}\sqrt{W}$,

$$\begin{aligned}2\sqrt{W}\frac{d}{dt}\sqrt{W} &\le -\sigma_\epsilon W \\ &\quad + \|u\|\sqrt{(1+\epsilon^2)(\|\dot q\|^2 + \|DV(q)\|^2)} \\ \Rightarrow \frac{d}{dt}\sqrt{W} &\le -\frac{1}{2}\sigma_\epsilon\sqrt{W} + \sqrt{\frac{1+\epsilon^2}{\mu_1\lambda_2(P_{1,\epsilon})}}\frac{\|u\|}{2},\end{aligned}$$

recalling from Lemma 1 that

$$\sigma_\epsilon = \frac{1}{\mu_2}\min_{z\in\mathbb{R}^2, z\ne 0}\frac{z^T Q_\epsilon z}{z^T P_{2,\epsilon} z}.$$

Consequently,

$$\begin{aligned}\sqrt{W(q(t),\dot q(t))} &\le e^{-\frac{\sigma_\epsilon}{2}t}(\sqrt{W_0} - \kappa_\epsilon U) + \kappa_\epsilon U \\ \Rightarrow W &\le e^{-\sigma_\epsilon t}(\sqrt{W_0} - \kappa_\epsilon U)^2 \\ &\quad + 2e^{-\frac{\sigma_\epsilon}{2}t}(\sqrt{W_0} - \kappa_\epsilon U)\kappa_\epsilon U \\ &\quad + \kappa_\epsilon^2 U^2,\end{aligned} \tag{55}$$

where $W_0 = W(q(0),\dot q(0))$, and $\kappa_\epsilon = \frac{1}{\sigma_\epsilon}\sqrt{\frac{1+\epsilon^2}{\mu_1\lambda_2(P_{1,\epsilon})}}$.

Using Lemma 1 and (49), $W(q(t),\dot q(t))$ estimates $E(t)$ via

$$\eta_{1,\epsilon}W(q(t),\dot q(t)) \le E(t) \le \eta_{2,\epsilon}W(q(t),\dot q(t)). \tag{56}$$

Therefore, if

$$\begin{aligned}c_1 &= \eta_{2,\epsilon}\big(\sqrt{\tfrac{E(0)}{\eta_{1,\epsilon}}} - \kappa_\epsilon U\big)^2 \\ c_2 &= 2\eta_{2,\epsilon}\kappa_\epsilon U\big(\sqrt{\tfrac{E(0)}{\eta_{1,\epsilon}}} - \kappa_\epsilon U\big),\end{aligned}$$

then

$$E(t) \le c_1 e^{-\sigma_\epsilon t} + c_2 e^{\frac{-\sigma_\epsilon}{2}t} + \eta_{2,\epsilon}\kappa_\epsilon^2 U^2 \tag{57}$$

for any ϵ with $0 < \epsilon < \epsilon_{\max}$. The proposition then follows from observing that $K = \min_{0<\epsilon<\epsilon_{\max}} \eta_{2,\epsilon}\kappa_\epsilon^2$.

Remark Proposition 1 by itself does not give an estimate for $q(t)$, but if V has a strict global minimum, then $V^{-1}(KU^2)$ gives an estimate of $q(t)$. Thus, if V has a strict global minimum, Proposition 1 establishes bouned input bounded state stability.

5.3 Application to Unmodeled Dynamics

Returning to the problem from §4, the following proposition gives a sufficient condition for the existence of a restriction of the state space of the modeled system (34) that would guarantee a bound on $(q_2, \dot q_2)$, and thus yield (35) and (36).

Recalling the system described by (32) and (33) from §4, let

$$\zeta(q_1,q_2) = \frac{\left|\|D_2\tilde V(q_1,q_2)\|^2 - \|D_2\tilde V(q_1,0)\|^2\right|}{V_2(q_2)}$$

for $(q_1,q_2) \in Q_1 \times Q_2$ such that $q_2 \ne 0$. For any $\alpha > 0$, let $Q_2' = \{q_2 \in Q_2 \mid V_2(q_2) \le K\alpha\}$, and let

$$\beta(q_1) = \sup_{q_2\in Q_2', q_2\ne 0}\zeta(q_1,q_2)$$

if the sup exists.

Define $Q_1' \subset Q_1$ by

$$\begin{aligned}Q_1' = \{q_1 \in Q_1 \mid &\beta(q_1)\in\mathbb{R}, \\ &\|D_2\tilde V(q_1,0)\|^2 < (1-\beta(q_1)K)\alpha\},\end{aligned} \tag{58}$$

where K is the constant from Proposition 1 for the mechanical system defined by M_2, V_2, and ϕ_2 from §4 (assume that the requirements in §5.1 hold for M_2, V_2, and ϕ_2). Note that if $q_2 \in Q_2'$, then

$$\|D_2\tilde V(q_1,q_2)\|^2 \le \beta(q_1)K\alpha + \|D_2\tilde V(q_1,0)\|^2. \tag{59}$$

Let $E_{20} = \frac{1}{2}\dot q_2(0)^T M_2(q_2(0))\dot q_2(0) + V_2(q_2(0))$, and δ be the value of ϵ where the minimum in Proposition 1 is achieved.

Proposition 2 *If Q_2' is compact, then for any compact $Q_1'' \subset Q_1'$ there exists $r > 0$ such that if $q_1(t) \in Q_1''$ and $\|\dot q_1(t)\| < r$ for all t, and $E_{20} \le \frac{\eta_{1,\delta}}{\eta_{2,\delta}}K\alpha$, then*

$$\frac{1}{2}\dot q_2(t)^T M_2(q_2(t))\dot q_2(t) + V_2(q_2(t)) \le K\alpha \tag{60}$$

for all t.

Proof From (33), (55) and (57), it suffices to confirm that $\frac{1}{2}\dot{q}_2^T M_2(q_2)\dot{q}_2 + V_2(q_2) \leq K\alpha$ implies $\|\tilde{C}_{21}\dot{q}_1 + D_2\tilde{V}\|^2 \leq \alpha$ to establish (60). Since $\|\tilde{C}_{21}\dot{q}_1\| \to 0$ as $\dot{q}_1 \to 0$, and $Q_1'' \times Q_2'$ is compact, the proposition follows if $\frac{1}{2}\dot{q}_2^T M_2(q_2)\dot{q}_2 + V_2(q_2) \leq K\alpha$ implies $\|D_2\tilde{V}\|^2 < \alpha$. This is guaranteed by (58) and (59).

Thus, if Q_1' is not empty, the total system (28) can be modeled using (34), (35), and (36) together with the additional restriction that $q \in Q_1''$ and $\|\dot{q}\| < r$.

6 CONCLUSIONS

This paper has presented a model reduction technique for systems comprised of Lagrangian subsystems with force interconnections based on Lyapunov stability, which guarantees estimates of the effects of the neglected dynamics. Future work includes examining the conservatism in the bounds produced, automating part of the reduction process and examining the implications of these reductions on control strategy selection.

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